Fractal structures of normal and anomalous diffusion in nonlinear nonhyperbolic dynamical systems

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A paradigmatic nonhyperbolic dynamical system exhibiting deterministic diffusion is the smooth nonlinear climbing sine map. We find that this map generates fractal hierarchies of normal and anomalous diffusive regions as functions of the control parameter. The measure of these self-similar sets is positive, parameter-dependent, and in case of normal diffusion it shows a fractal diffusion coefficient. By using a Green-Kubo formula we link these fractal structures to the nonlinear microscopic dynamics in terms of fractal Takagi-like functions.

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It is well-known that diffusion processes in nonlinear dynamical systems may be generated by microscopic deterministic chaos in the equations of motion. This fact points to a somewhat deeper foundation of nonequilibrium statistical mechanics than modeling diffusive transport by stochastic random walks. In order to understand deterministic diffusion, statistical mechanics was suitably combined with dynamical systems theory [1–3]. Much was learned by analyzing simple models of deterministic transport such as one- and two-dimensional maps [4–7], chaotic billiards in external fields [8], periodic Lorentz gases [9,10], and certain differential equations [11]. However, characteristics of deterministic diffusion were also observed experimentally, that is, in dissipative systems driven by periodic forces such as Josephson junctions in the presence of microwave radiation [12], in superionic conductors [13], and in systems exhibiting charge-density waves [14]. The equations of motion of these systems are generally of the form of some nonlinear pendulum equation. In the limiting case of strong dissipation, these differential equations were reduced to nonhyperbolic one-dimensional maps sharing certain symmetries [15]. The so-called climbing sine map is a well-known example of this class of maps [4,5]. Due to its nonhyperbolicity, the map posseses a rich dynamics consisting of chaotic diffusive motion, ballistic dynamics, and localized orbits. Under parameter variation these different types of dynamics are highly intertwined resulting in complicated scenarios related to the appearance of periodic windows [5]. On the other hand, for simple one-dimensional hyperbolic maps it was shown that the diffusion coefficient is typically a fractal function of control parameters [16,17]. An analogous behavior was also detected for other transport coefficients [18], and in more complicated models [8–10]. However, up to now the fractality of transport coefficients could be assessed for hyperbolic systems only, whereas, to our knowledge, the fractal nature of classical transport coefficients in the broad class of nonhyperbolic systems was not discussed.

Here we show that the nonhyperbolicity of the climbing sine map does not destroy these fractal characteristics of deterministic diffusive transport. On the contrary, fractal structures appear for normal diffusive parameters as well as for anomalous diffusive regions. We argue that higher-order memory effects are crucial to understand the origin of these fractal hierarchies in nonhyperbolic systems. By using a Green-Kubo formula for diffusion, the dynamical correlations are recovered in terms of fractal Takagi-like functions. We furthermore show that the distribution of periodic windows forms devil's staircase-like structures as a function of the parameter and that the complementary sets of chaotic dynamics have a positive measure in parameter space that increases by increasing the parameter value.

The climbing sine map we study is defined as

$$x_{n+1} = M_a(x_n) \ , \quad M_a(x) := x + a\sin(2\pi x) \ , \tag{1}$$

where $a \in \mathbb{R}$ is a control parameter, $x \in \mathbb{R}$, and x_n is the position of a point particle at discrete time n. Obviously, $M_a(x)$ possesses translation and reflection symmetry,

$$M_a(x+m) = M_a(x) + m \ , \quad M_a(-x) = -M_a(x) \ . \tag{2}$$

The periodicity of the map naturally splits the phase space into different cells $(m, m+1]$, $m \in \mathbb{Z}$. We will be interested in parameters $a > 0.732644$ for which the extrema of the map exceed the boundaries of each cell for the first time indicating the onset of diffusive motion.

The bifurcation diagram of the associated reduced map $\tilde{M}_a(\tilde{x}) := M_a(x) \bmod 1$, $\tilde{x} := x \ mod\ 1$, consists of infinitely many periodic windows, see Fig. 1. Whenever there is a window the dynamics of Eq. (1) is either ballistic or localized [5]. Fig. 1 demonstrates that this scenario has a strong impact on the diffusion coefficient defined by $D(a) := \lim_{n\to\infty} \langle x_n^2 \rangle/(2n)$, where the brackets denote an ensemble average over moving particles. For localized dynamics orbits are confined within some finite interval in phase space implying subdiffusive behavior for which the diffusion coefficient vanishes, whereas for ballistic motion particles propagate superdiffusively with the diffusion coefficient being proportional to n. Only for normal

diffusion $D(a)$ is nonzero and finite. At the boundaries of each periodic window there is transient intermittent-like behavior eventually resulting in normal diffusion with $D(a) \sim a^{(\pm\frac{1}{2})}$ [4,5]. Here we are interested in the complete parameter-dependent diffusion coefficient. For this purpose we compute $D(a)$ from numerical simulations by using the Green-Kubo formula for maps [3,10,16,18],

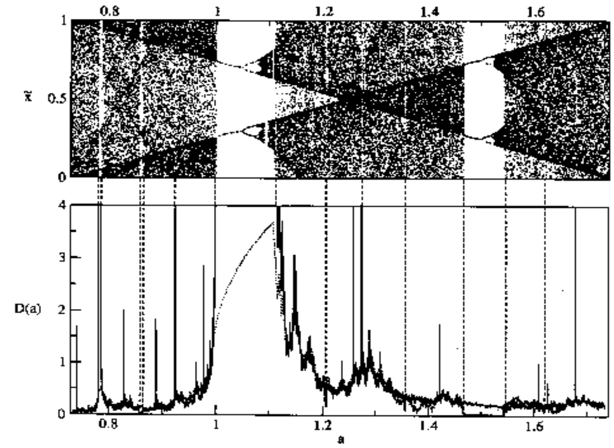

FIG. 1. Upper panel: bifurcation diagram for the climbing sine map. Lower panel: diffusion coefficient computed from simulations as a function of the control parameter a in comparison with the correlated random walk approximation $D^1_{10}(a)$ (dots). The dashed vertical lines connect regions of anomalous diffusion, $D(a) \to \infty$ or $D(a) \to 0$, with ballistic and localized dynamics in respective windows of the bifurcation diagram.

$$D_n(a) = \langle j_a(\tilde{x}_0) J^n_a(\tilde{x})\rangle - \frac{1}{2}\langle j^2_a(\tilde{x}_0)\rangle \ , \qquad (3)$$

where the angular brackets denote an average over the invariant density of the reduced map, $\langle \ldots \rangle := \int d\tilde{x}\rho(\tilde{x}) \ldots$. The jump velocity j_a is defined by $j_a(\tilde{x}_n) := [x_{n+1}] - [x_n] \equiv [M_a(\tilde{x}_n)]$, where the square brackets denote the largest integer less than the argument. The sum $J^n_a(\tilde{x}) := \sum_{k=0}^{n} j_a(\tilde{x}_k)$ gives the integer value of the displacement of a particle after n time steps that started at some initial position $x \equiv x_0$, and we call it jump velocity function. Eq. (3) defines a time-dependent diffusion coefficient which, in case of normal diffusion, converges to $D(a) \equiv \lim_{n\to\infty} D_n(a)$. In our simulations we truncated $J^n_a(\tilde{x})$ after having obtained enough convergence for $D(a)$, that is, after 20 time steps. The invariant density was obtained by solving the continuity equation for $\rho(\tilde{x})$ with the histogram method of Ref. [1].

The highly non-trivial behavior of the diffusion coefficient in Fig. 1 can qualitatively be understood as follows: The Green-Kubo formula Eq. (3) splits the dynamics into an inter-cell dynamics, in terms of integer jumps, and into an intra-cell dynamics, as represented by the invariant density. We first approximate the invariant density in Eq. (3) to $\rho(\tilde{x}) \simeq 1$ irrespective of the fact that it is a complicated function of x and a [5]. This approximate diffusion coefficient we denote with a superscript in Eq. (3), $D^1_n(a)$. The term for $n = 0$ is well-known as the stochastic random walk approximation for maps, which excludes any higher-order correlations [4,5,17]. The generalization $D^1_n(a)$, $n > 0$ was called correlated random walk approximation [10]. We now use this systematic expansion to analyze the diffusion coefficient of the climbing sine map in terms of higher-order correlations.

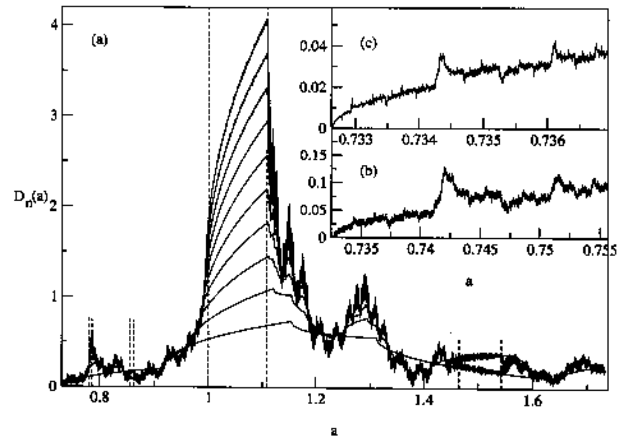

FIG. 2. (a) Sequence of correlated random walks $D^1_n(a)$ for $n = 1, \cdots, 10$. Note the quick convergence for normal diffusive parameters. The dashed lines define the same periodic windows as in Fig.1. The inserts (b) and (c) contain blowups of $D^1_{10}(a)$ in the initial region of (a). They show self-similar behavior on smaller and smaller scales.

In Fig.2 (a) we depict results for $D_n(a)$ at $n = 1, \ldots, 10$. One clearly observes convergence of this approximation in parameter regions with normal diffusion. Indeed, a comparison of $D^1_{10}(a)$ with $D(a)$, as shown in Fig.1, demonstrates that there is qualitative agreement on large scales. On the other hand, for parameters corresponding to ballistic motion the sequence of $D^1_n(a)$ diverges, in agreement with $D(a) \to \infty$, whereas for localized dynamics it alternates between two solutions. This oscillation is reminiscent of the dynamical origin of localization in terms of certain period-two orbits. That these solutions are non-zero is due to the fact that the invariant density was approximated. In regions of normal diffusion this approximation nicely reproduces the irregularities in the diffusion coefficient. Even more importantly, the magnifications in Fig.2 give clear evidence for a self-similar structure of the diffusion coefficient. Our results thus show that dealing with correlated jumps only yields a qualitative understanding of normal and anomalous diffusion in the climbing sine map.

We now further analyze the dynamical origin of these different structures. According to its definition, the time-dependent jump velocity function $J^n_a(\tilde{x})$ fulfills the recursion relation

$$J^n_a(\tilde{x}) = j_a(\tilde{x}) + J^{n-1}_a(M_a(\tilde{x})) \ . \qquad (4)$$

$J^n_a(\tilde{x})$ is getting extremely complicated after some time

steps, thus we introduce the more well-behaved function

$$T_a^n(\tilde{x}) := \int_0^x J_a^n(z)\, dz, \quad T_a^n(0) \equiv T_a^n(1) \equiv 0. \qquad (5)$$

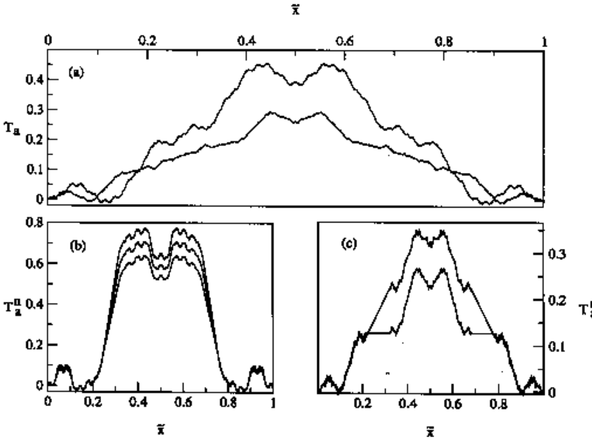

FIG. 3. Functions $T_a^n(\tilde{x})$ for the climbing sine map representing the number of jumps particles starting at $\tilde{x}$ have traveled when this number is integrated from zero to $\tilde{x}$, see Eqs. (4)-(7). (a) depicts diffusive dynamics at $a = 1.2397$ (upper curve) and at $a = 1.7427$ (lower curve), (b) ballistic dynamics at $a = 1.0$, and (c) localized dynamics at $a = 1.5$. In (a) the limiting case for $n \to \infty$ is shown, in (b) and (c) it was $n = 5, 6, 7$. Note the divergence in (b) and the oscillation in (c).

Integration of Eq. (4) thus yields the recursive functional equation

$$T_a^n(\tilde{x}) = t_a(\tilde{x}) + \frac{1}{\tilde{M}_a(\tilde{x})}\, T_a^{n-1}(\tilde{M}_a(\tilde{x})) - I(\tilde{x}) \qquad (6)$$

with the integral term

$$I(\tilde{x}) := \int_0^{\tilde{M}_a(\tilde{x})} dz g''(z) T_a^{n-1}(z) \;, \qquad (7)$$

where $t_a(\tilde{x}) := \int dz\, j_a(z)$, and $g''(z)$ is the second derivative of the inverse function of $\tilde{M}_a(\tilde{x})$ [19]. For piecewise linear hyperbolic maps $I(\tilde{x})$ simply disappears and the derivative in front of the second term reduces to the local slope of the map thus recovering ordinary de Rham-type equations [3,17,18]. It is not known to us how to directly solve this generalized de Rham-equation for the climbing sine map, however, solutions can alternatively be constructed from Eq. (5) on the basis of simulations. Results are shown in Fig. 3. For normal diffusive parameters the limit $T_a(\tilde{x}) = \lim_{n\to\infty} T_a^n(\tilde{x})$ exists, and the respective curve is fractal over the whole unit interval somewhat resembling (generalized) fractal Takagi functions [3,17,18]. However, in case of periodic windows $T_a^n(\tilde{x})$ either diverges due to ballistic flights, or it oscillates indicating localization. Interestingly, in these functions the corresponding attracting sets appear in form of smooth, nonfractal regions on fine scales, whereas the other regions appear to be fractal.

The diffusion coefficient can now be formulated in terms of these fractal functions by integrating Eq. (3). For $a \in (0.732644, 1.742726]$ we get

$$D(a) = 2\left[T_a(\tilde{x}_2)\rho(\tilde{x}_2) - T_a(\tilde{x}_1)\rho(\tilde{x}_1)\right] - D_0^\rho(a), \qquad (8)$$

where $\tilde{x}_i,\ i = 1, 2$, is defined by $[M_a(\tilde{x}_i)] := 1$, and $D_0^\rho(a) := \int_{\tilde{x}_1}^{\tilde{x}_2} d\tilde{x}\rho(\tilde{x})$. Our previous approximation $D_n^1(a)$ is recovered from this equation as a special case.

The intimate relation between periodic windows and the irregular behavior of the diffusion coefficient motivates us to investigate the structure of the periodic windows in the climbing sine map in more detail. The appearance of windows was analyzed quite extensively for non-diffusive unimodal maps [20], whereas for diffusive maps on the line, apart from the preliminary studies of Refs. [5], nothing appears to be known. The windows are generated by certain periodic orbits, consequently there are infinitely many of them, and they are believed to be dense in the parameter set [2]. Windows with ballistic dynamics are born through tangent bifurcations, further undergo Feigenbaum-type scenarios and eventually terminate at crisis points. Windows with localized orbits only occur at even periods. They start with tangent bifurcations and exhibit a symmetry breaking at slope-type bifurcation points.

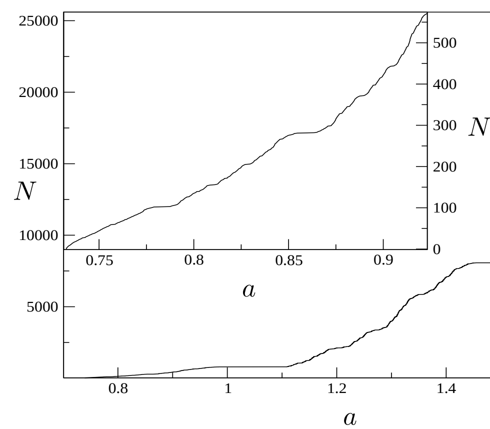

FIG. 4. Devil's staircase-like structure formed by the distribution of periodic windows as a function of the control parameter. N is the integrated number of period six-windows. The inset shows a blowup of the initial region.

In order to analyze the structure of the regions of anomalous diffusion, we sum up the number of period six-windows as a function of the parameter, that is, the total number is increased by one for any parameter value at which a new period six-window appears. This sum forms a devil's staircase-like structure in parameter space indicating an underlying Cantor set-like distribution for the corresponding anomalous diffusive region, see Fig. 4. The (Lebesque) measure of periodic windows is obviously positive, hence this set must be a fat fractal [21]. Its self-similar structure can quantitatively be assessed by computing the so-called fatness exponent [22]. We are

furthermore interested in the parameter dependence of this fractal structure, therefore we divide the parameter line into subsets labeled by the integer value of the map maximum on the unit interval, $[M_a(x_{max})] = j$, $j \in \mathbb{Z}$. For $j = 1, 2, 3$ we obtain a fatness exponent of 0.45 with errors of 0.03, 0.04, and 0.05 for the different j. We mention that this value was conjectured to be universal and was also obtained for non-diffusive unimodal maps [22].

We now study the measure of the windows as a function of the parameter. For this purpose we computed all windows up to period six for the first subset, up to period 5 for $j = 2, 3$, and we summed up their measures in the respective subsets. We find that the total measure decays exponentially as a function of j while oscillating with odd and even values of j on a finer scale [23]. This oscillation can be traced back to windows generated by localized dynamics, which only appear at even periods thus contributing only periodically to the total measure. However, different measures of 'ballistic' and 'localized' windows decay with the same rate. We have furthermore computed the complementary measure C_j of diffusive dynamics in the jth subset of parameters. We find that $C_1 = 0.783$, $C_2 = 0.808$, and $C_3 = 0.932$ with an error of ± 0.002, so the measure of the diffusive regions is always non-zero and seems to approach one with increasing parameter values.

We finally remark that the climbing sine map is of the same functional form as the respective nonlinear equation in the two-dimensional standard map, which is considered to be a standard model for many physical, Hamiltonian dynamical systems. Indeed, both models are motivated by the driven pendulum, both are strongly nonhyperbolic, and although the standard map is area-preserving it too exhibits a highly irregular parameter-dependent diffusion coefficient. Understanding the origin of these irregularities was the subject of intensive research [2,7], however, so far the complexity of the system prevented to reveal its possibly fractal nature. A suitably adapted version of our approach to nonhyperbolic diffusive dynamics may enable to make some progress in this direction.

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